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On the low-Reynolds-number flow in a helical pipe

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A non-orthogonal helical co-ordinate system is introduced to study the effect of curvature and torsion on the flow in a helical pipe. It is found that both curvature and torsion induce non-negligible effects when the Reynolds number is less than about 40. When the Reynolds number is of order unity, torsion induces a secondary flow consisting of one single recirculating cell while curvature causes an increased flow rate. These effects are quite different from the two recirculating cells and decreased flow rate at high Reynolds numbers.

1. Introduction

Fluid flow in helical pipes occurs in many engineering processes, particularly those involving viscometry or convective heat transfer. Dean (1927, 1928) first studied the flow in a curved pipe using a concentric toroidal co-ordinate system. Numerous authors subsequently utilized Dean's co-ordinates to investigate the various aspects of the flow in a toroidal pipe.

Can the results from the analysis of a toroidal pipe be applied to a helical pipe? Helical pipes involve both a curvature and a torsion or twist. None of the two dozen plus existing theoretical papers on the flow in curved pipes considered non-zero torsion. Only a few papers (theoretically, Topakoglu 1967 and Larrain & Bonilla 1970; numerically, Truesdell & Adler 1970 and Austin & Seader 1973) consider the effect of non-zero curvature. The other papers, including Dean's, study the flow in the limit of zero curvature.

The present work attempts to determine whether curvature and torsion should be considered in the flow in a helical pipe through the introduction of a non-orthogonal, helical co-ordinate system.

2. Navier–Stokes equations with one axis following a space curve

Let one axis be described by

$$\mathbf{R} = X(s)\,\mathbf{i} + Y(s)\,\mathbf{j} + Z(s)\,\mathbf{k}, \tag{1}$$

where s represents arc length along this axis and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors in Cartesian directions. The tangent $\mathbf{T}$, normal $\mathbf{N}$ and binormal $\mathbf{B}$ can be defined by

$$\mathbf{T} = \frac{d\mathbf{R}}{ds}, \quad \mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}, \quad \mathbf{B} = \mathbf{T}\times\mathbf{N}. \tag{2}$$

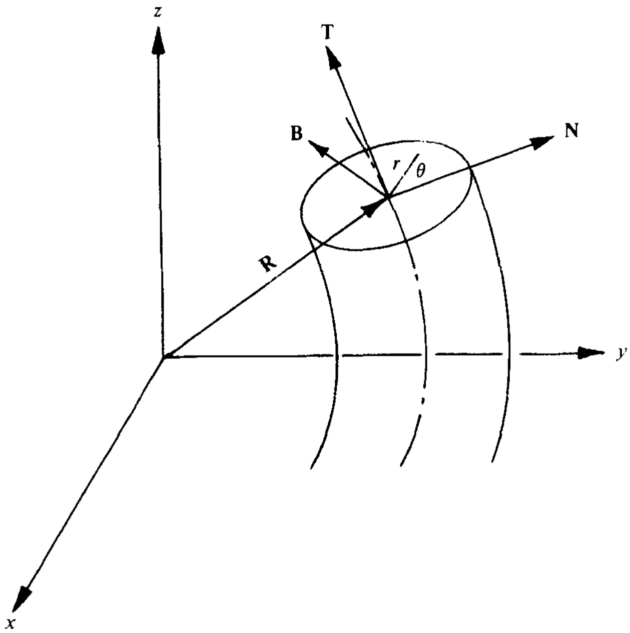


FIGURE 1. The co-ordinate system.

Here κ is the curvature, and **T**, **N**, **B** are orthogonal unit vectors. The Frenet formulae give

$$\frac{d\mathbf{N}}{ds} = \tau\mathbf{B} - \kappa\mathbf{T}, \quad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}, \tag{3}$$

where τ is the torsion. We construct a new co-ordinate system (r, θ, s) such that any Cartesian position vector **x** can be expressed as

$$\mathbf{x} = \mathbf{R}(s) + r\cos\theta\mathbf{N}(s) + r\sin\theta\mathbf{B}(s) \tag{4}$$

(see figure 1). Using equations (2, 3, 4), we obtain

$$d\mathbf{x}.d\mathbf{x} = (dr)^2 + r^2(d\theta)^2 + [(1-\kappa r\cos\theta)^2 + \tau^2 r^2](ds)^2 + 2\tau r^2\,ds\,d\theta. \tag{5}$$

The description of any point in this system is unique for $r \leqslant \kappa^{-1}$. Notice that the last term in (5) indicates *non-orthogonality* of (r, θ, s). If torsion τ is zero the system reduces to the *orthogonal* co-ordinates of Dean (1927) and Murata, Miyake & Inaba (1976). From equation (5), the metric tensors are found to be

$$\left.\begin{array}{c} g_{11} = g^{11} = 1, \quad g_{22} = r^2, \quad g^{22} = G/r^2M, \\ g_{33} = G, \quad g^{33} = 1/M, \quad g_{23} = \tau r^2, \quad g^{23} = -\tau/M, \\ g_{12} = g^{12} = g_{13} = g^{13} = 0, \end{array}\right\} \tag{6}$$

$$G \equiv (1-\kappa r\cos\theta)^2 + \tau^2 r^2, \quad M \equiv (1-\kappa r\cos\theta)^2. \tag{7}$$

The non-zero Christoffel symbols are

$$\left.\begin{array}{l} \Gamma^1_{22} = -r, \quad \Gamma^1_{23} = -\tau r, \quad \Gamma^1_{33} = -\dfrac{1}{2}\dfrac{\partial G}{\partial r}, \quad \Gamma^2_{21} = \dfrac{1}{r}, \\ \Gamma^2_{13} = \dfrac{\tau}{M}\left(\dfrac{G}{r} - \dfrac{1}{2}\dfrac{\partial G}{\partial r}\right), \quad \Gamma^2_{23} = -\dfrac{\tau}{2M}\dfrac{\partial G}{\partial \theta}, \quad \Gamma^2_{33} = \dfrac{G}{r^2M}\left(\dfrac{d\tau}{ds}r^2 - \dfrac{1}{2}\dfrac{\partial G}{\partial \theta}\right) - \dfrac{\tau}{2M}\dfrac{\partial G}{\partial s}, \\ \Gamma^3_{13} = -\dfrac{r\tau^2}{M} + \dfrac{1}{2M}\dfrac{\partial G}{\partial r}, \quad \Gamma^3_{23} = \dfrac{1}{2M}\dfrac{\partial G}{\partial \theta}, \quad \Gamma^3_{33} = -\dfrac{\tau}{M}\left(\dfrac{d\tau}{ds}r^2 - \dfrac{1}{2}\dfrac{\partial G}{\partial \theta}\right) + \dfrac{1}{2M}\dfrac{\partial G}{\partial s}. \end{array}\right\} \tag{8}$$

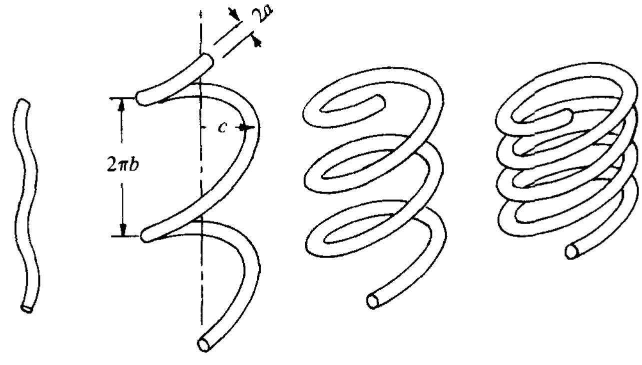


FIGURE 2. Helical coils with $\kappa a = 0{\cdot}1$. (*a*) $\tau a = 0{\cdot}3$, (*b*) $\tau a = 0{\cdot}05$, (*c*) $\tau a = 0{\cdot}02$, (*d*) $\tau a = 0{\cdot}01$.

The Navier–Stokes equations and the continuity equation in tensorial form are

$$\frac{\partial v^i}{\partial t} + v^j\left(\frac{\partial v^i}{\partial x^j} + \Gamma^i_{\alpha j} v^\alpha\right) = -\frac{g^{ij}}{\rho}\frac{\partial p}{\partial x^j} + \nu g^{kj} v^i_{,jk}, \tag{9}$$

$$\frac{\partial v^i}{\partial x^i} + \Gamma^i_{\alpha i} v^\alpha = 0. \tag{10}$$

Here the covariant derivative is

$$v^i_{,jk} \equiv \frac{\partial^2 v^i}{\partial x^k\,\partial x^j} + \Gamma^i_{j\alpha}\frac{\partial v^\alpha}{\partial x^k} + \Gamma^i_{\alpha k}\frac{\partial v^\alpha}{\partial x^j} - \Gamma^\alpha_{jk}\frac{\partial v^i}{\partial x^\alpha} + \left(\frac{\partial \Gamma^i_{j\alpha}}{\partial x^k} + \Gamma^i_{\beta k}\Gamma^\beta_{j\alpha} - \Gamma^\beta_{jk}\Gamma^i_{\beta\alpha}\right) v^\alpha, \tag{11}$$

with the summation convention implied. Let

$$x^1 = r, \quad x^2 = \theta, \quad x^3 = s, \tag{12}$$

and (u, v, w) be the physical velocity components in (r, θ, s) respectively. Then the tensorial velocity components v^i are related by

$$v^1 = u, \quad v^2 = v/r, \quad v^3 = w/\sqrt{G}. \tag{13}$$

3. Formation of the equations of the flow in a helical pipe

Let the s axis be described by the helix

$$\mathbf{R} = c\cos\frac{s}{(b^2+c^2)^{\frac{1}{2}}}\,\mathbf{i} + c\sin\frac{s}{(b^2+c^2)^{\frac{1}{2}}}\,\mathbf{j} + \frac{bs}{(b^2+c^2)^{\frac{1}{2}}}\,\mathbf{k}, \tag{14}$$

where b and c are constants. Then

$$\kappa = \frac{c}{b^2+c^2}, \quad \tau = \frac{b}{b^2+c^2}. \tag{15}$$

Figure 2 shows the general shapes of helical pipes with the same radius a and the same curvature κ. Only when torsion τ is zero do we obtain the toroidal pipes studied by previous authors.

We shall make the following assumptions:

(1) The pipe is long enough for the end effects to be ignored. We expect the velocities are independent of s and the pressure to be a linear function of s. Suppose the mean

pressure gradient $d\bar{p}/ds$ is given. We can then define a velocity scale U by

$$U \equiv \frac{a^2}{4\mu}\left(-\frac{d\bar{p}}{ds}\right). \tag{16}$$

(2) Both curvature and torsion are small, i.e.

$$\kappa a \equiv \epsilon \ll 1, \quad \tau a \equiv \epsilon\lambda \ll 1, \tag{17}$$

where λ is a constant of order unity. These assumptions include a wide variety of realistic helical pipes.

(3) The flow is steady and laminar. The Reynolds number, defined as $R \equiv Ua/\nu$, is order unity. As we shall see later, the influences of curvature and torsion are most prominent at low Reynolds numbers. Low-Reynolds-number flow occurs in viscometry that uses coiled capillaries.

Notice that, in the case of a straight circular pipe, our R becomes the usual Reynolds of number based on mean velocity and diameter. The Dean number K is defined by

$$K = \left(\frac{d\bar{p}}{ds}\right)^2 \frac{a^7\kappa}{8\mu^2\nu^2} = 2\epsilon R^2. \tag{18}$$

The dependent and independent variables are then normalized as follows:

$$\eta \equiv r/a, \quad x \equiv s/a, \tag{19}$$

$$w = U(w_0 + \epsilon w_1 + \epsilon^2 w_2 + \ldots), \tag{20}$$

$$u = U(\epsilon u_1 + \epsilon^2 u_2 + \ldots), \quad v = U(\epsilon v_1 + \epsilon^2 v_2 + \ldots), \tag{21}$$

$$p = (\mu U/a)(p_0 + \epsilon p_1 + \epsilon^2 p_2 + \ldots). \tag{22}$$

Here we have utilized the fact that the axial flow w_0 dominates for pipes of small curvature and torsion. The Navier–Stokes equation (9) and continuity equation (10) are then perturbed. The boundary conditions are that the velocities are zero on $\eta = 1$ and that the mean pressure gradient is held constant. The primary flow is found to be Poiseuille flow in a straight tube:

$$w_0 = 1 - \eta^2, \quad p_0 = -4x + \text{constant}. \tag{23}$$

4. The secondary flow

Without going into the details, the order-ϵ terms of equations (9)–(10) are

$$R\cos\theta w_0^2 = -\frac{\partial p_1}{\partial \eta} + \frac{\partial^2 u_1}{\partial \eta^2} + \frac{1}{\eta^2}\left(\frac{\partial^2 u_1}{\partial \theta^2} - 2\frac{\partial v_1}{\partial \theta} + \eta\frac{\partial u_1}{\partial \eta} - u_1\right), \tag{24}$$

$$-R\sin\theta\frac{w_0^2}{\eta} = -\left(\frac{1}{\eta^2}\frac{\partial p_1}{\partial \theta} - \lambda\frac{\partial p_0}{\partial x}\right) + \left(\frac{\partial^2}{\partial \eta^2} + \frac{3}{\eta}\frac{\partial}{\partial \eta} + \frac{1}{\eta^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{v_1}{\eta}\right) + \frac{2\lambda}{\eta}\frac{\partial w_0}{\partial \eta} + \frac{2}{\eta^3}\frac{\partial u_1}{\partial \theta}, \tag{25}$$

$$Ru_1\frac{\partial w_0}{\partial \eta} = -\left(\frac{\partial p_1}{\partial x} + 2\eta\cos\theta\frac{\partial p_0}{\partial x}\right) + \left(\frac{\partial^2}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial}{\partial \eta} + \frac{1}{\eta^2}\frac{\partial^2}{\partial \theta^2}\right)(\eta\cos\theta w_0 + w_1) - 3\cos\theta\frac{\partial w_0}{\partial \eta}, \tag{26}$$

$$\frac{\partial u_1}{\partial \eta} + \frac{1}{\eta}\frac{\partial v_1}{\partial \theta} + \frac{u_1}{\eta} = 0. \tag{27}$$

We set
$$u_1 = R\cos\theta\frac{f(\eta)}{\eta}, \quad v_1 = -R\sin\theta\frac{df}{d\eta} + \lambda g(\eta), \tag{28}$$

$$p_1 = R\cos\theta P(\eta), \tag{29}$$

and substitute into (24)–(27). The boundary conditions are

$$f(1) = f'(1) = g(1) = 0 \tag{30}$$

and f, g, P bounded at $\eta = 0$. The solution is

$$f = \tfrac{1}{288}(\eta^7 - 6\eta^5 + 9\eta^3 - 4\eta), \tag{31}$$

$$g = \eta^3 - \eta,$$

$$P = -\tfrac{1}{12}(2\eta^5 - 6\eta^3 + 9\eta), \tag{32}$$

$$w_1 = -\tfrac{3}{4}(\eta^3 - \eta)\cos\theta - \frac{R^2\cos\theta}{11\,520}(\eta^9 - 10\eta^7 + 30\eta^5 - 40\eta^3 + 19\eta). \tag{33}$$

If a stream function ψ is defined as

$$u_1 = \frac{1}{\eta}\frac{\partial\psi}{\partial\theta}, \quad v_1 = -\frac{\partial\psi}{\partial\eta},$$

we find
$$\frac{\psi}{R} = \frac{\sin\theta}{288}(\eta^7 - 6\eta^5 + 9\eta^3 - 4\eta) - \frac{1}{4}\left(\frac{\lambda}{R}\right)(\eta^4 - 2\eta^2 + 1). \tag{34}$$

The first term on the right-hand side of equation (34) represents the two symmetric recirculating cells found by Dean (1927). The other term due to the effect of torsion is a general rotation motion. This term is important when λ/R or $\tau/(\kappa R)$ is not negligible. In fact the torsion term is so dominant that the two recirculating cells become one cell when $\lambda/R \geqslant 1/24 = 0{\cdot}04167$. Figure 3 shows the effect of torsion on the secondary flow.

5. The effect on flow rate

Since the co-ordinate system is not orthogonal the velocity w, in general, is not perpendicular to the θ, η plane. One must be careful in the calculation of flow rate. The covariant component of the velocity perpendicular to the θ, η plane is

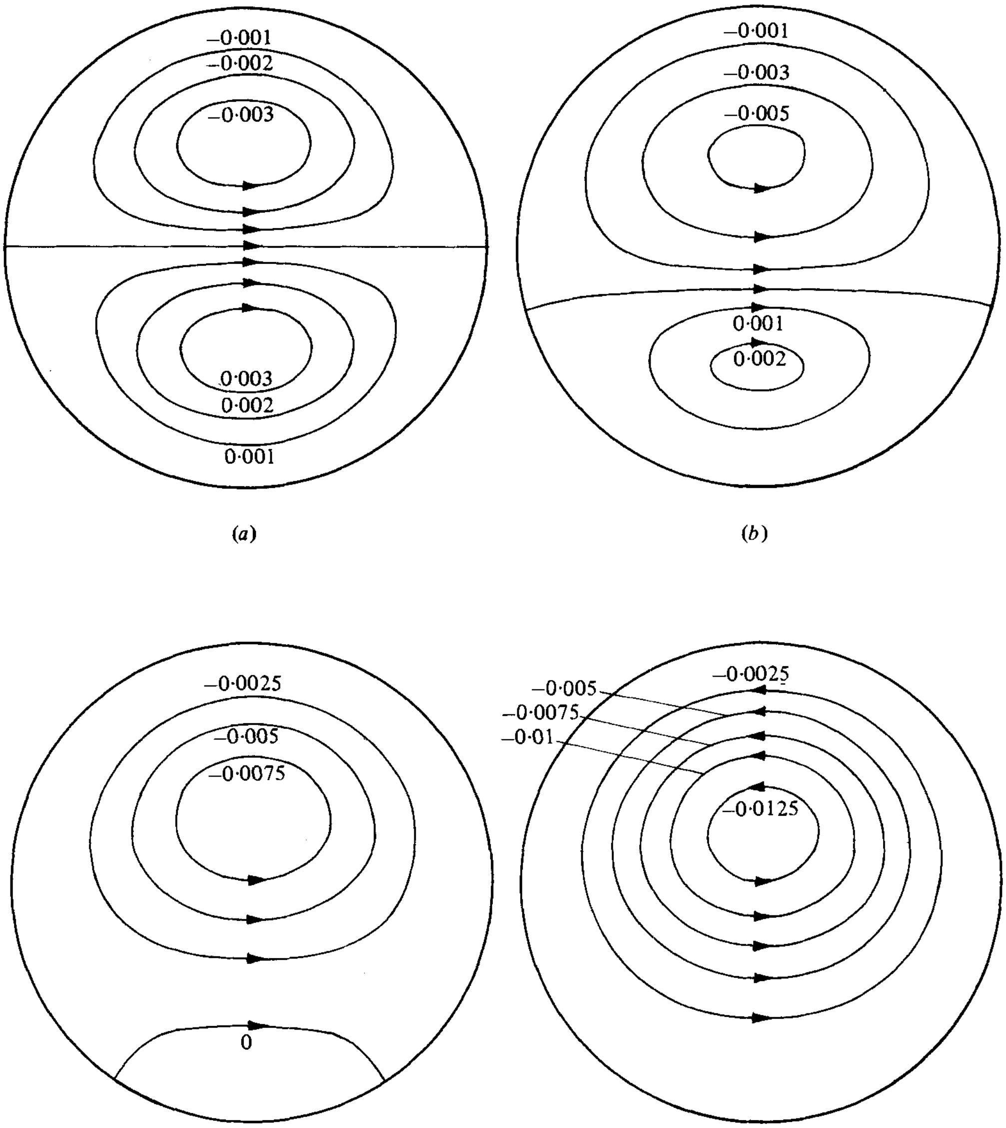
$$\sqrt{(g^{33})}\, g_{3i} v^i = \frac{1}{\sqrt{M}}(\tau r v + \sqrt{G}\, w). \tag{35}$$

The flow Q is thus

$$Q = \int_0^{2\pi}\int_0^a \sqrt{g^{33}}\, g_{3i} v^i r\, dr\, d\theta = Ua^2 \int_0^{2\pi}\int_0^1 [w_0 + \epsilon w_1 + \epsilon^2(\lambda\eta v_1 + \tfrac{1}{2}\lambda^2\eta^2 w_0 + w_2)]\,\eta\, d\eta\, d\theta. \tag{36}$$

The flow w_1 is periodic in θ, and does not contribute to the flow rate. From equations (9), (10) we obtain the second-order equations:

$$R\left(u_1\frac{\partial u_1}{\partial\eta} + \frac{v_1}{\eta}\frac{\partial u_1}{\partial\theta} - \frac{v_1^2}{\eta} - 2\lambda v_1 w_0 - \eta\lambda^2 w_0^2 + 2w_0 w_1\cos\theta + \eta w_0^2\cos^2\theta\right)$$
$$= -\frac{\partial p_2}{\partial\eta} + \frac{\partial^2 u_2}{\partial\eta^2} + \frac{1}{\eta^2}\left[\frac{\partial^2 u_2}{\partial\theta^2} - 2\frac{\partial v_2}{\partial\theta} - 2\lambda\eta\frac{\partial}{\partial\theta}(\eta\cos\theta w_0 + w_1) + \eta\frac{\partial u_2}{\partial\eta} - u_2\right]; \tag{37}$$



(c) (d)

FIGURE 3. Secondary flow. (a) $\lambda/R = 0$, (b) $\lambda/R = 0{\cdot}01$, (c) $\lambda/R = 0{\cdot}03$, (d) $\lambda/R = 0{\cdot}05$. Values of constant ψ/R are shown.

$$R\left[u_1\frac{\partial}{\partial\eta}\left(\frac{v_1}{\eta}\right)+\frac{v_1}{\eta}\frac{\partial}{\partial\theta}\left(\frac{v_1}{\eta}\right)+\frac{2u_1v_1}{\eta^2}+2\lambda\frac{u_1w_0}{\eta}-2\sin\theta\frac{w_0w_1}{\eta}-\sin\theta\cos\theta w_0^2\right]$$
$$=-\frac{1}{\eta^2}\left(\frac{\partial p_2}{\partial\theta}+\lambda^2\eta^2\frac{\partial p_0}{\partial\theta}\right)+\left(\frac{\partial^2}{\partial\eta^2}+\frac{3}{\eta}\frac{\partial}{\partial\eta}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{v_2}{\eta}\right)-\left(\cos\theta\frac{\partial}{\partial\eta}-\frac{\sin\theta}{\eta}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{\eta}\right)\left(\frac{v_1}{\eta}\right)$$
$$+\frac{2}{\eta^3}\frac{\partial u_2}{\partial\theta}+\sin\theta\frac{u_1}{\eta^2}+2\lambda\cos\theta\frac{\partial w_0}{\partial\eta}+\frac{2\lambda}{\eta}\frac{\partial}{\partial\eta}(\eta\cos\theta w_0+w_1); \quad (38)$$

$$R\left[\left(u_1\frac{\partial}{\partial\eta}+\frac{v_1}{\eta}\frac{\partial}{\partial\theta}\right)(\eta\cos\theta w_0+w_1)+u_2\frac{\partial w_0}{\partial\eta}-2u_1w_0\cos\theta+2u_1w_0\sin\theta+\lambda\eta w_0^2\sin\theta\right]$$
$$=-3\eta^2\cos^2\theta\frac{\partial p_0}{\partial x}+\lambda\frac{\partial p_1}{\partial\theta}+\left(\frac{\partial^2}{\partial\eta^2}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{\eta}\frac{\partial}{\partial\eta}\right)(w_2+\eta\cos\theta w_1-\tfrac{1}{2}\eta^2\lambda^2w_0+\eta^2w_0\cos^2\theta)$$
$$-3\eta\cos^2\theta\frac{\partial w_0}{\partial\eta}+\left(\frac{3\sin\theta}{\eta}\frac{\partial}{\partial\theta}-3\cos\theta\frac{\partial}{\partial\eta}\right)(\eta\cos\theta w_0+w_1); \tag{39}$$

$$\frac{\partial u_2}{\partial\eta}+\frac{1}{\eta}\frac{\partial v_2}{\partial\theta}+\frac{u_2}{\eta}-\cos\theta\,u_1+\sin\theta\,v_1=0. \tag{40}$$

Let the non-periodic (in θ) component be denoted by a bar. Without going into the details, the solutions to (37)–(40) are

$$\bar{u}_2=\frac{R}{2}f(\eta),\quad \bar{v}_2=0, \tag{41}$$

$$\bar{w}_2=\frac{\eta^2\lambda^2}{2}(1-\eta^2)-\tfrac{1}{32}(11\eta^4-14\eta^2+3)$$
$$+\frac{R^2}{230400}(-7\eta^{10}+75\eta^8-200\eta^6+175\eta^4+105\eta^2-148)$$
$$-\frac{R^4}{6635520}\left(\frac{\eta^{16}}{16}-\tfrac{8}{7}\eta^{14}+\tfrac{33}{4}\eta^{12}-\tfrac{157}{5}\eta^{10}+\tfrac{569}{8}\eta^8-99\eta^6+\tfrac{331}{4}\eta^4-38\eta^2+\tfrac{4119}{560}\right). \tag{42}$$

The effect of torsion appears as the first term on the right-hand side of (42). But, when we substitute (23), (28) and (42) into (36), all the torsion terms cancel. The flow rate is found to be

$$\frac{Q}{Q_s}=1-\frac{\epsilon^2}{48}\left[\frac{1541}{67200}\left(\frac{R}{6}\right)^4+\frac{11}{10}\left(\frac{R}{6}\right)^2-1\right]+O(\epsilon^4), \tag{43}$$

where Q_s is the flow rate if the pipe were straight. To the order considered, torsion does not affect flow rate. Equation (43) agrees with the result for a toroidal tube obtained by Topakoglu (1967) and Larrain & Bonilla (1970). Notice also that, owing to the non-zero curvature, the flow rate cannot be expressed in terms of Dean number alone.

6. Discussion

The helical co-ordinate system was first suggested by Nicholson (1910) who erroneously thought the system was orthogonal. For orthogonal co-ordinates, one can easily derive the governing equations by using simple scale factors (Batchelor 1970). But, for a non-orthogonal system as presented in this paper, tensor analysis is necessary to derive the governing equations.

We see that the major effect of curvature is on the flow rate while the major effect of torsion is on the secondary flow. These effects must be considered when the Reynolds number is low. For example, suppose we have a typical helical pipe of $\kappa a = 0{\cdot}1$ and $\tau a = 0{\cdot}02$. Our analysis of the secondary flow shows asymmetry of the two recirculating cells and relatively stagnant regions become appreciable when R is less than 20 (Dean number less than 80). The two-celled structure is completely destroyed when R is less than 4·8 (Dean number less than 4·6). Since the secondary flow is quite important in transport processes, the effect of torsion must be considered at low Reynolds numbers.

To order ϵ^2, torsion does not affect the flow rate. Curvature, however, has appreciable effect at low Reynolds numbers. Equation (43) shows the R^2 term is comparable to the R^4 term when R is less than 40 (Dean number less than 320 if $\epsilon = 0{\cdot}1$). In fact, in the limit of small ϵ, when $R < 5{\cdot}67$ the flow rate in a curved pipe is *larger* than that of a straight pipe. This fact was also predicted by Larrain & Bonilla (1970) for the toroidal pipe.

The assumptions on curvature and torsion parameters are $\kappa a = \epsilon \ll 1$ and $\tau a = \epsilon\lambda \ll 1$. As seen from figure 2, these assumptions are well satisfied for all practical purposes. Our assumption that the Reynolds number R is of order unity needs some comment, since higher-order terms involve increasingly higher powers of R and this may limit R to be less than one. However, the range of validity for our expansion can be extended to much larger R values. This is because the constant coefficients (e.g. in equation (43)) are very small. Using a computer, Larrain & Bonilla (1970) calculated the resistance coefficient for the non-twisted toroidal pipe as a polynomial up to R^{28} and found that the series converged for Dean number (defined in equation (18)) smaller than 576. Since the helical pipe is similar, this gives $R < 17\sqrt{\epsilon}$ ($R < 53$ for $\epsilon = 0{\cdot}1$) as the range of convergence. Computer-calculated extended series have also been discussed by Van Dyke (1978), who showed that inferences of the flow at much higher Reynolds numbers can sometimes be drawn. We expect, however, the effects of curvature and torsion would diminish as the Reynolds number is increased.

Appendix. The Navier–Stokes equations in general intrinsic co-ordinates

The continuity equation is

$$\frac{\partial u}{\partial r}+\frac{\partial}{\partial \theta}\left(\frac{v}{r}\right)+\frac{\partial}{\partial s}\left(\frac{w}{J}\right)+(\Gamma^2_{12}+\Gamma^3_{12})\,u+\Gamma^3_{23}\frac{v}{r}+(\Gamma^3_{33}+\Gamma^2_{23})\frac{w}{J}=0. \tag{44}$$

Here $J \equiv \sqrt{G}$ and the Christoffel symbols are from equation (8). The momentum equations are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{v}{r}\left(\frac{\partial u}{\partial \theta}+\Gamma^1_{22}\frac{v}{r}+\Gamma^1_{23}\frac{w}{J}\right)+\frac{w}{J}\left(\frac{\partial u}{\partial s}+\Gamma^1_{23}\frac{v}{r}+\Gamma^1_{33}\frac{w}{J}\right)$$
$$=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(v^1_{,11}+\frac{G}{r^2M}v^1_{,22}+\frac{1}{M}v^1_{,33}-\frac{2\tau}{M}v^1_{,23}\right); \tag{45}$$

$$\frac{\partial}{\partial t}\left(\frac{v}{r}\right)+u\left[\frac{\partial}{\partial r}\left(\frac{v}{r}\right)+\Gamma^2_{12}\frac{v}{r}+\Gamma^2_{13}\frac{w}{J}\right]+\frac{v}{r}\left[\frac{\partial}{\partial \theta}\left(\frac{v}{r}\right)+\Gamma^2_{12}u+\Gamma^2_{23}\frac{w}{J}\right]$$
$$+\frac{w}{J}\left[\frac{\partial}{\partial s}\left(\frac{v}{r}\right)+\Gamma^2_{13}u+\Gamma^2_{23}\frac{v}{r}+\Gamma^2_{33}\frac{w}{J}\right]$$
$$=-\frac{1}{\rho}\left(\frac{G}{r^2M}\frac{\partial p}{\partial \theta}-\frac{\tau}{M}\frac{\partial p}{\partial s}\right)+\nu\left(v^2_{,11}+\frac{G}{r^2M}v^2_{,22}+\frac{1}{M}v^2_{,33}-\frac{2\tau}{M}v^2_{,23}\right); \tag{46}$$

$$\frac{\partial}{\partial t}\left(\frac{w}{J}\right)+u\left[\frac{\partial}{\partial r}\left(\frac{w}{J}\right)+\Gamma^3_{13}\frac{w}{J}\right]+\frac{v}{r}\left[\frac{\partial}{\partial \theta}\left(\frac{w}{J}\right)+\Gamma^3_{23}\frac{w}{J}\right]+\frac{w}{J}\left[\frac{\partial}{\partial s}\left(\frac{w}{J}\right)+\Gamma^3_{13}u+\Gamma^3_{23}\frac{v}{r}+\Gamma^3_{33}\frac{w}{J}\right]$$
$$=-\frac{1}{\rho}\left(\frac{-\tau}{M}\frac{\partial p}{\partial \theta}+\frac{1}{M}\frac{\partial p}{\partial s}\right)+\nu\left(v^3_{,11}+\frac{G}{r^2M}v^3_{,22}+\frac{1}{M}v^3_{,33}-\frac{2\tau}{M}v^3_{,23}\right); \tag{47}$$

where

$$v^1_{,11} = \frac{\partial^2 u}{\partial r^2}, \tag{48}$$

$$v^1_{,22} = \frac{\partial^2 u}{\partial \theta^2} + 2\Gamma^1_{22}\frac{\partial}{\partial \theta}\left(\frac{v}{r}\right) + 2\Gamma^1_{23}\frac{\partial}{\partial \theta}\left(\frac{w}{J}\right) - \Gamma^1_{22}\frac{\partial u}{\partial r} + \Gamma^1_{22}\Gamma^2_{21}u + (\Gamma^1_{22}\Gamma^2_{23} + \Gamma^1_{23}\Gamma^3_{23})\frac{w}{J}; \tag{49}$$

$$\begin{aligned} v^1_{,33} = {} & \frac{\partial^2 u}{\partial s^2} + 2\Gamma^1_{23}\frac{\partial}{\partial s}\left(\frac{v}{r}\right) + 2\Gamma^1_{33}\frac{\partial}{\partial s}\left(\frac{w}{J}\right) - \Gamma^1_{33}\frac{\partial u}{\partial r} - \Gamma^2_{33}\frac{\partial u}{\partial \theta} - \Gamma^3_{33}\frac{\partial u}{\partial s} + (\Gamma^1_{23}\Gamma^2_{13} + \Gamma^1_{33}\Gamma^3_{13})u \\ & + \left(\frac{\partial}{\partial s}\Gamma^1_{23} + \Gamma^1_{23}\Gamma^2_{23} + \Gamma^1_{33}\Gamma^3_{23} - \Gamma^2_{33}\Gamma^1_{22} - \Gamma^3_{33}\Gamma^1_{23}\right)\frac{v}{r} + \left(\frac{\partial}{\partial s}\Gamma^1_{33}\right)\frac{w}{J}; \end{aligned} \tag{50}$$

$$\begin{aligned} v^1_{,23} = {} & \frac{\partial^2 u}{\partial \theta\, \partial s} + \Gamma^1_{22}\frac{\partial}{\partial s}\left(\frac{v}{r}\right) + \Gamma^1_{23}\frac{\partial}{\partial s}\left(\frac{w}{J}\right) + \Gamma^1_{23}\frac{\partial}{\partial \theta}\left(\frac{v}{r}\right) + \Gamma^1_{33}\frac{\partial}{\partial \theta}\left(\frac{w}{J}\right) \\ & - \Gamma^1_{23}\frac{\partial u}{\partial r} - \Gamma^2_{23}\frac{\partial u}{\partial \theta} - \Gamma^3_{23}\frac{\partial u}{\partial s} - (\Gamma^2_{23}\Gamma^1_{22} + \Gamma^3_{23}\Gamma^1_{23})\frac{v}{r} + \left(\frac{\partial}{\partial s}\Gamma^1_{23}\right)\frac{w}{J}; \end{aligned} \tag{51}$$

$$\begin{aligned} v^2_{,11} = {} & \frac{\partial^2}{\partial r^2}\left(\frac{v}{r}\right) + 2\Gamma^2_{12}\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + 2\Gamma^2_{13}\frac{\partial}{\partial r}\left(\frac{w}{J}\right) + \left(\frac{\partial}{\partial r}\Gamma^2_{12} + \Gamma^2_{12}\Gamma^2_{12}\right)\frac{v}{r} \\ & + \left(\frac{\partial}{\partial r}\Gamma^2_{13} + \Gamma^2_{12}\Gamma^2_{13} + \Gamma^2_{13}\Gamma^3_{13}\right)\frac{w}{J}; \end{aligned} \tag{52}$$

$$\begin{aligned} v^2_{,22} = {} & \frac{\partial^2}{\partial \theta^2}\left(\frac{v}{r}\right) + 2\Gamma^2_{12}\frac{\partial u}{\partial \theta} + 2\Gamma^2_{23}\frac{\partial}{\partial \theta}\left(\frac{w}{J}\right) - \Gamma^1_{22}\frac{\partial}{\partial r}\left(\frac{v}{r}\right) \\ & + \left(\frac{\partial}{\partial \theta}\Gamma^2_{23} + \Gamma^2_{12}\Gamma^1_{23} + \Gamma^2_{23}\Gamma^3_{23} - \Gamma^1_{22}\Gamma^2_{13}\right)\frac{w}{J}; \end{aligned} \tag{53}$$

$$\begin{aligned} v^2_{,33} = {} & \frac{\partial^2}{\partial s^2}\left(\frac{v}{r}\right) + 2\Gamma^2_{13}\frac{\partial u}{\partial s} + 2\Gamma^2_{23}\frac{\partial}{\partial s}\left(\frac{v}{r}\right) + 2\Gamma^2_{33}\frac{\partial}{\partial s}\left(\frac{w}{J}\right) \\ & - \Gamma^1_{33}\frac{\partial}{\partial r}\left(\frac{v}{r}\right) - \Gamma^2_{33}\frac{\partial}{\partial \theta}\left(\frac{v}{r}\right) - \Gamma^3_{33}\frac{\partial}{\partial s}\left(\frac{v}{r}\right) + \left(\frac{\partial}{\partial s}\Gamma^2_{33}\right)\frac{w}{J} \\ & + \left(\frac{\partial}{\partial s}\Gamma^2_{13} + \Gamma^2_{23}\Gamma^2_{13} + \Gamma^2_{33}\Gamma^3_{13} - \Gamma^2_{33}\Gamma^2_{12} - \Gamma^3_{33}\Gamma^2_{13}\right)u \\ & + \left(\frac{\partial}{\partial s}\Gamma^2_{23} + \Gamma^2_{13}\Gamma^1_{23} + \Gamma^2_{23}\Gamma^2_{23} + \Gamma^2_{33}\Gamma^3_{23} - \Gamma^1_{33}\Gamma^2_{12} - \Gamma^3_{33}\Gamma^2_{23}\right)\frac{v}{r}; \end{aligned} \tag{54}$$

$$\begin{aligned} v^2_{,23} = {} & \frac{\partial^2}{\partial \theta\, \partial s}\left(\frac{v}{r}\right) + \Gamma^2_{12}\frac{\partial u}{\partial s} + \Gamma^2_{23}\frac{\partial}{\partial s}\left(\frac{w}{J}\right) + \Gamma^2_{13}\frac{\partial u}{\partial \theta} + \Gamma^2_{23}\frac{\partial}{\partial \theta}\left(\frac{v}{r}\right) + \Gamma^2_{33}\frac{\partial}{\partial \theta}\left(\frac{w}{r}\right) \\ & - \Gamma^1_{23}\frac{\partial}{\partial r}\left(\frac{v}{r}\right) - \Gamma^2_{23}\frac{\partial}{\partial \theta}\left(\frac{v}{r}\right) - \Gamma^3_{23}\frac{\partial}{\partial s}\left(\frac{v}{r}\right) - \Gamma^3_{23}\Gamma^2_{13}u \\ & + (\Gamma^2_{13}\Gamma^1_{22} - \Gamma^1_{23}\Gamma^2_{12} - \Gamma^3_{23}\Gamma^2_{23})\frac{v}{r} + \left(\frac{\partial}{\partial s}\Gamma^2_{23}\right)\frac{w}{J}; \end{aligned} \tag{55}$$

$$v^3_{,11} = \frac{\partial^2}{\partial r^2}\left(\frac{w}{J}\right) + 2\Gamma^3_{13}\frac{\partial}{\partial r}\left(\frac{w}{J}\right) + \left(\frac{\partial}{\partial r}\Gamma^3_{13} + \Gamma^3_{13}\Gamma^3_{13}\right)\frac{w}{J}; \tag{56}$$

$$v^3_{,22} = \frac{\partial^2}{\partial \theta^2}\left(\frac{w}{J}\right) + 2\Gamma^3_{23}\frac{\partial}{\partial \theta}\left(\frac{w}{J}\right) - \Gamma^1_{22}\frac{\partial}{\partial r}\left(\frac{w}{J}\right) + \left(\frac{\partial}{\partial \theta}\Gamma^3_{23} + \Gamma^3_{23}\Gamma^3_{23} - \Gamma^1_{22}\Gamma^3_{13}\right)\frac{w}{J}; \tag{57}$$

$$v^3_{,33} = \frac{\partial^2}{\partial s^2}\left(\frac{w}{J}\right) + 2\Gamma^3_{13}\frac{\partial u}{\partial s} + 2\Gamma^3_{32}\frac{\partial}{\partial s}\left(\frac{v}{r}\right) + 2\Gamma^3_{33}\frac{\partial}{\partial s}\left(\frac{w}{J}\right) - \Gamma^1_{33}\frac{\partial}{\partial r}\left(\frac{w}{J}\right)$$
$$- \Gamma^2_{33}\frac{\partial}{\partial \theta}\left(\frac{w}{J}\right) - \Gamma^3_{33}\frac{\partial}{\partial s}\left(\frac{w}{J}\right) + \left(\frac{\partial}{\partial s}\Gamma^3_{13} + \Gamma^3_{23}\Gamma^2_{13}\right)u$$
$$+ \left(\frac{\partial}{\partial s}\Gamma^3_{23} + \Gamma^3_{13}\Gamma^1_{23} + \Gamma^3_{23}\Gamma^2_{23}\right)\frac{v}{r} + \left(\frac{\partial}{\partial s}\Gamma^3_{33}\right)\frac{w}{J}; \qquad (58)$$

$$v^3_{,23} = \frac{\partial^2}{\partial s\,\partial \theta}\left(\frac{w}{J}\right) + \Gamma^3_{23}\frac{\partial}{\partial s}\left(\frac{w}{J}\right) + \Gamma^3_{13}\frac{\partial u}{\partial \theta} + \Gamma^3_{23}\frac{\partial}{\partial \theta}\left(\frac{v}{r}\right) + \Gamma^3_{33}\frac{\partial}{\partial \theta}\left(\frac{w}{J}\right)$$
$$- \Gamma^1_{23}\frac{\partial}{\partial r}\left(\frac{w}{J}\right) - \Gamma^2_{23}\frac{\partial}{\partial \theta}\left(\frac{w}{J}\right) - \Gamma^3_{23}\frac{\partial}{\partial s}\left(\frac{w}{J}\right) + \left(\frac{\partial}{\partial s}\Gamma^3_{23}\right)\frac{w}{J}$$
$$+ (\Gamma^3_{23}\Gamma^2_{12} - \Gamma^3_{23}\Gamma^3_{13})\,u + (\Gamma^3_{13}\Gamma^1_{22} - \Gamma^3_{23}\Gamma^3_{23})\frac{v}{r}. \qquad (59)$$

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